Physics on the Back of a Cocktail Napkin

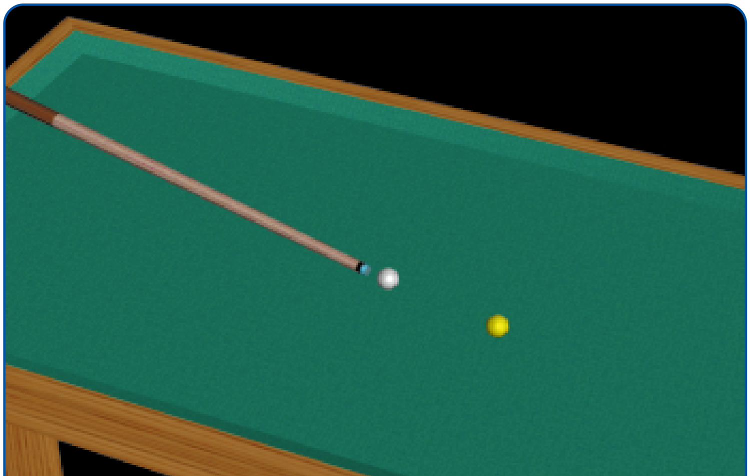
I am generally pretty disciplined about working normal hours during the week. However, on Fridays I like to shoot pool and eat pizza at the local sports bar. Since happy hour starts at three, I sometimes need to move work down the street. I was working out how to win a serious game of nine ball when one of those geeky discussions broke out about pool table physics. Pool, like many sports, is dominated by the laws of physics. Good players have an excellent sense of the application of force, the physics of collisions, and the influence of friction on objects in motion.

Last month I described how friction could be used to increase the realism of the physics model in real-time games. The demo program made it possible to see how various coefficients affected a mass-and-spring model. However, it wasn't very much fun. In order to demonstrate how a solid physical foundation can actually create interesting game play, I need to pull some of these concepts together into a real application. A pool table simulation is a natural choice. It will allow me to apply many of the techniques I have covered as well as provide some ideas that can be converted easily to other sports such as golf or tennis.

Two Ball, Corner Pocket

In order to understand pool, I need to understand collisions between billiard balls. Fortunately, billiard balls are all spheres of equal size and weight. That makes the collision calculations a bit easier. Let me begin by looking at the frictionless case. I suspect that if I ignore the ball's rotation and do not consider friction, the ball collision will behave exactly like a particle at the ball's center of mass. That would be great, as I could use the code from a previous column. However, I want to make sure. Figure 1 shows a typical collision.

Ball A is moving at a speed of 20 feet per second and collides with ball B at a 40 degree angle. In order to determine the velocity of each ball after the collision, I need to apply dynamics. A collision between two rigid bodies, which



Like most things in life, pool provides an excellent venue for a physics lesson.

v	Velocity of body (vector)
m	Mass of a body
ε	Coefficient of restitution
n	Line of collision or collision normal
t	Line tangent to collision
j	Impulse force
N	Force normal to surface
g	Gravitational force
μ_S	Coefficient of sliding friction
μ_R	Coefficient of sliding friction
R	Radius of the ball
I	Inertia tensor
ω	Angular velocity
α	Angular acceleration
r	Vector from center of mass to point of contact

TABLE 1. *A summary of the notation used in this article.*

When not wasting time at the pub eating hot wings and shooting pool, Jeff can be found at Darwin 3D. There he creates real-time 3D graphics for a variety of applications. Drop him a line at jeffl@darwin3d.com.

occurs in a very short time and during which the two bodies exert relatively large forces on each other, is called an impact. The force between these two bodies during the collision is called an impulsive force, symbolized by j. The common normal to the surfaces of the bodies in contact is called the line of collision, n in Figure 1.

The first step is to break the initial velocity of ball A into its components along the line of collision, n, and the tangent to the collision, t.

$$v_A = 20 \text{ ft / sec}$$
$$(v_A \cdot n) = v_A \cos(40) = 15.32 \text{ ft / sec}$$
$$(v_A \cdot t) = v_A \sin(40) = -12.86 \text{ ft / sec}$$

The impulsive force acting during the collision is directed along the line of collision. Therefore, the t component of the velocity of each ball is not changed.

$$(v'_A \cdot t) = -12.86 \text{ ft / sec}$$
$$(v'_B \cdot t) = 0 \text{ ft / sec}$$

In order to determine the new velocity along the line of collision, I need to look at the impulsive force between the bodies. The impulse acts on both bodies at the same time. You may remember Newton's third law of motion, the forces exerted by two particles on each other are equal in magnitude and opposite in direction.

Since the impulse forces are equal and opposite, momentum is conserved before and after the collision. Remember, momentum of a rigid body is mass times velocity (mv).

$$m_A(v_A \cdot n) + m_B(v_B \cdot n) = m_A(v'_A \cdot n) + m_B(v'_B \cdot n)$$
$$m(15.32) + m(0) = m_A(v'_A \cdot n) + m_B(v'_B \cdot n)$$
$$(v_A \cdot n) + (v_B \cdot n) = 15.32 \quad \text{(Eq. 1)}$$

This equation can't be solved without some more information. In my March 1999 column ("Collision Response: Bouncy, Trouncy, Fun"), I discussed the coefficient of restitution.

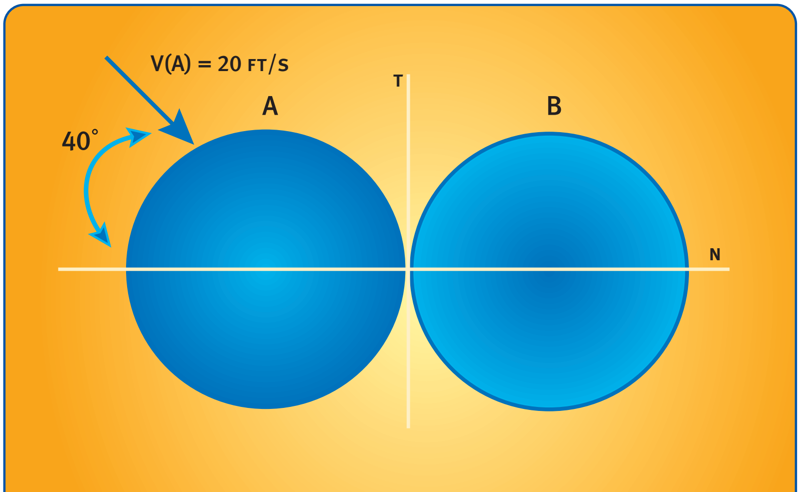


FIGURE 1. *A simple collision between two billiard balls.*

This is the scalar value between 0 and 1 relating the velocities of bodies before and after a collision via the formula:

$$(v'_B \cdot n) - (v'_A \cdot n) = \varepsilon[(v_A \cdot n) - (v_B \cdot n)] \quad \text{(Eq. 2)}$$

For this example, I'm using a coefficient of restitution ε of 0.8. I can use this formula to create a second equation.

$$(v'_B \cdot n) - (v'_A \cdot n) = \varepsilon[(v_A \cdot n) - (v_B \cdot n)]$$
$$(v'_B \cdot n) - (v'_A \cdot n) = 0.8[(15.32) + (0)]$$
$$(v'_B \cdot n) - (v'_A \cdot n) = 12.26 \quad \text{(Eq. 3)}$$

Solving Equations 1 and 3, I get the velocities of the two billiard balls after the collision.

$$v'_A = (1.53, -12.86) \text{ ft / sec}$$
$$v'_B = (13.79, 0.0) \text{ ft / sec}$$

In order to solve this problem in the simulation, I need to derive the impulse force directly. The impulse force creates a change in momentum of the two bodies with the following relationship.

$$m_A v_A + jn = m_A v'_A$$
$$v'_A = v_A + \frac{j}{m_A} n$$
$$m_B v_B - jn = m_B v'_B$$
$$v'_B = v_B - \frac{j}{m_B} n \quad \text{(Eq. 4)}$$

These formulas can be combined with Equation 2 to determine the impulse force given the relative velocity and the coefficient of restitution.

$$j = \frac{-(1+\varepsilon) v_{AB} \cdot n}{n \cdot n \left(\frac{1}{m_A} + \frac{1}{m_B} \right)} \quad \text{(Eq. 5)}$$

You can plug Equation 5 back into the example problem and make sure it works. Remember that because of Newton's third law, the impulse is equal and opposite for the two colliding bodies. When you apply Equation 5 to the B ball, remember to negate it.

Those of you who read Chris Hecker's column on collision response ("Physics, Part 3: Collision Response," Behind the Screen, February/March 1997) will recognize Equation 4 as the impulse equation for a general body that does not rotate. When we do not consider the rotation of the billiard balls, they behave exactly like the particles used in my March 1999 mass-and-spring demo. My suspicion was correct, and I can use the particle dynamics system as a base for the demo.

For many applications, this would probably be more than enough to get a nice physical simulation. In fact, I imagine many pool simulations end right there.

This level of simulation is probably enough for other games, such as pinball. However, anyone who has played much pool knows that this is not the end of the story. The rotation of the ball caused by the reaction with the table makes a tremendous difference in the realism of the simulation.

Slip Sliding Away

When a billiard ball is hit with the cue stick, the ball starts moving across the table. If the ball is struck along its center of mass, the ball is not initially rotating.

However, soon the ball starts rolling along. Friction between the ball and the table causes this roll to occur. You can see this situation in Figure 2.

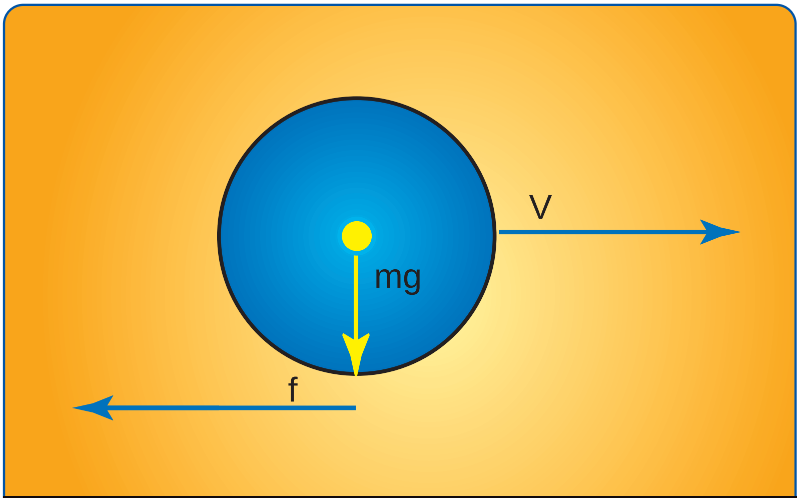


FIGURE 2. *A billiard ball sliding across the table.*

The ball is traveling with the forward velocity, v. In last month's column ("The Trials and Tribulations of Tribology," Graphic Content, August 1999), I discussed the use of kinetic friction via the Coulomb dry friction model. I am going to call this force "sliding friction." The force of friction applied to a body sliding over a surface is given by the following formula:

$$f = \mu_R N = \mu_s mg \qquad \text{(Eq. 6)}$$

The friction force is applied in the direction opposite the velocity. Since this force is applied to the surface of the ball and not its center of mass, the frictional force causes angular acceleration in the ball. As the ball rolls across the table, the angular velocity increases because of this sliding friction force. This continues until a time of equilibrium is reached, where the velocity of the point contacting the table equals the velocity of the center of mass. At this time, the ball is no longer sliding and is now rolling on the table. This situation is called a natural roll or rolling without sliding. In mathematical terms, this situation happens when

$$v = R\omega \qquad \text{(Eq. 7)}$$

where v is the velocity of the ball, ω is the angular velocity of the ball, and R is the ball's radius.

Now I need to show how the angular acceleration actually changes. This is going to mean bringing up another term, the inertia tensor, or I. You may remember from Chris Hecker's column on 3D physics ("Physics, Part 4: The Third Dimension," Behind the Screen, June 1997) that the inertia tensor relates the angular velocity of a body to the angular momentum of that body. For arbitrarily complex objects, creating the inertia tensor can be quite difficult. However, for a uniform sphere where the density is uniform across the sphere, it's quite easy. The inertia tensor for a sphere is

$$I = \frac{2mR^2}{5}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(Eq. 8)}$$

Therefore, the product of this matrix with any vector is a simple scaling of that vector. The relationship between the angular acceleration and the friction force then becomes

$$\alpha = \frac{r \leftrightarrow f}{I}$$

$$\alpha = \frac{r \leftrightarrow f}{\frac{2}{5}mR^2} = \frac{5(r \leftrightarrow f)}{2mR^2} \qquad \text{(Eq. 9)}$$

If I now take a look at the problem in Figure 2, I can calculate how long it will take for the ball to achieve natural roll given an initial velocity v. From the principle of impulse and momentum, I know some information about the linear momentum and angular velocity of the ball at a later time.

$$mv' = mv - f(\Delta t)$$

$$\omega' = \frac{f(\Delta t)r}{I} = \frac{5f(\Delta t)}{2mr}$$

In other words, the momentum at some later time is the initial momentum minus the impulse created by the friction force, f. I know the friction force from Equation 6.

$$mv' = mv - \mu_s mg(\Delta t)$$

$$v' = v - \mu_s g(\Delta t)$$

$$\omega' = \frac{5\mu_s g(\Delta t)}{2r}$$

At the point of natural roll, I know the state of equilibrium between angular and linear velocity from Equation 7.

$$v' = r\omega'$$

$$r\left[\frac{5\mu_s g(\Delta t)}{2r}\right] = v - \mu_s g(\Delta t)$$

$$\left(\frac{5\mu_s g}{2} + \mu_s g\right)(\Delta t) = v$$

$$\Delta t = \frac{2v}{7\mu_s g}$$

So you can see that as a result of the friction force of the table, a sliding billiard ball will always reach a point where it is rolling without sliding on the table. This is the type of realism I want to have in the simulation. A ball when struck should slide across the table, slowly settling to a state where it is rolling without slipping.

How Do I Stop This Crazy Thing?

One glaring problem remains. I can run the simulation with all of the physics discussed so far. When struck hard, a billiard ball will slide and then roll. Once the ball has reached this natural roll, there is nothing in my simulation that will keep it from continuing to roll forever. The friction force is gone since the point of contact is not moving relative to the table. I need to add another force that will slow down a rolling ball. I can add another frictional force, called rolling friction, which is applied when the ball is in natural roll. The form of rolling friction is

$$f_R = \mu_R mg$$

It is applied exactly like the sliding friction whenever the natural roll conditions apply. It is important to note that the coefficients of rolling and sliding friction are not necessarily the same. Think of a ball moving on a rubber surface. The coefficient of sliding friction would be very high. However, the rolling friction would be comparatively low, allowing the ball to roll across the surface easily.

Rack 'Em Up

Using these techniques I have created a demonstration of a simple pool table. The simulation uses rolling and sliding friction to simulate the way a real billiard ball moves across a table. Collision between balls is handled through conservation of momentum and the elastic collision model. There are several areas that still need work. I didn't talk about applying "English" to the shot by striking the ball off the center of mass. This is what makes shots such as a Masse, draw, or topspin possible. This is largely just a matter of where the impulse from the cue stick is applied. Also, the lack of friction between colliding balls does not allow effects such as collision induced spin.

Alas, that will have to wait for another time. Until then, see if you can modify the source code to handle these effects. Get the source and executable off the *Game Developer* web site at http://www.gdmag.com. ■

FOR FURTHER INFO

- Beer, Ferdinand and E. Russell Johnston. *Vector Mechanics for Engineers: Statics and Dynamics*, Sixth Ed. New York: WCB/McGraw-Hill, 1997.
- Hecker, Chris. "Behind the Screen," *Game Developer* (October/November 1996–June 1997). Also available on Chris's web site, http://www.d6.com.